Determinants and inverses

In this section, we give some properties of the determinants of invertible matrices, and then use determinants to give a formula for the solution of $A\vec{x} = \vec{b}$ when A is invertible, without calculating A^{-1}!

Product Theorem: If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Notice that if A is invertible, this tells us that

$$\det(A)\det(A^{-1}) = \det(I) = 1.$$

so $\det(A) \neq 0$, and $\det(A^{-1}) = \frac{1}{\det(A)}$.

Conversely, if A is a matrix and $\det A \neq 0$, we can take A to its RRE form R via elem. operations:

$$R = E_k E_{k-1} \cdots E_2 E_1 A.$$

so $$\det R = \underbrace{\det E_k \cdots \det E_2 \det E_1}_{\neq 0} \underbrace{\det A}_{\neq 0}$$

So $\det R \neq 0$, which means it doesn't have a row of zeros, so $R = I \Rightarrow A^{-1} = (E_k E_{k-1} \cdot \cdots \cdot E_1)$, so A is invertible.

We've thus proven the following:

Theorem: A is invertible if and only if $\det A \neq 0$.

Ex: If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3c & 0 & 1 \end{bmatrix}$, for which values of c is A invertible? i.e. when is $\det A \neq 0$?

Taking $R_3 - 3cR_1$, we get

$$\det A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0-3c(2) & 1-3c(-1) \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -6c & 1+3c \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & 2 \\ -6c & 1+3c \end{vmatrix} = (1+3c) - 2(-6c) = 1 + 3c + 12c = 1 + 15c$$

so $\det A = 0$ when $1 + 15c = 0$, i.e. when $c = \frac{-1}{15}$.

Thus, A is invertible when $c \neq \frac{-1}{15}$.

Thm: $\det A^T = \det A$.

Def: A square matrix is <u>orthogonal</u> if $A^{-1} = A^T$.

What can we say about the determinant of orthogonal matrices? If A is orthogonal, then

$$I = AA^{-1} = AA^T,$$

so $1 = \det I = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$

so $\det A = \pm 1$.

Adjugates

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we defined the adjugate to be $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For an $n\times n$ matrix, there is a simple description of the adjugate. First define the <u>cofactor matrix</u> of A to be $[c_{ij}(A)]$. i.e. the (i,j)-entry is the corresponding cofactor.

Def: The <u>adjugate</u> of A is the transpose of the cofactor matrix. That is,

$$\operatorname{adj}(A) = [c_{ij}(A)]^T.$$

Ex: If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$, the cofactor matrix is

$$\begin{bmatrix} \begin{vmatrix} 3 & 1 \\ 7 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 4 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 4 & 7 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 7 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -10 & 4 & -12 \\ 2 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

so the adjugate is $\operatorname{adj}(A) = \begin{bmatrix} -10 & 2 & 2 \\ 4 & -1 & -1 \\ -12 & 1 & 3 \end{bmatrix}$

Just like in the 2×2 case, we can use the adjugate to find the inverse:

Adjugate Formula: If A is a square matrix, then

$$A(\operatorname{adj} A) = (\det A) I.$$

Thus, if $\det A \neq 0$, the inverse of A is

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Warning: Do <u>NOT</u> use this formula to compute the inverse. This formula is for theoretical purposes only.

We can also use it to compute a single entry of the inverse of a matrix:

Ex: If $A = \begin{bmatrix} 4 & 7 & 9 \\ 1 & 0 & 5 \\ 2 & 2 & -8 \end{bmatrix}$, what is the $(3,2)$-entry of A^{-1}?

$$\det A = \begin{vmatrix} 4 & 7 & 9 \\ 1 & 0 & 5 \\ 2 & 2 & -8 \end{vmatrix} = \begin{vmatrix} 4 & 7 & -11 \\ 1 & 0 & 0 \\ 2 & 2 & -18 \end{vmatrix} = -1 \begin{vmatrix} 7 & -11 \\ 2 & -18 \end{vmatrix} = -(-126+22) = 104$$

column 3 − 5 (column 1)

cofactor expansion along row 2

The $(3,2)$-entry of A^{-1} will be the $(2,3)$-entry of the cofactor matrix divided by $\det A$.

So the $(3,2)$-entry is

$$\frac{1}{104} C_{23}(A) = \frac{1}{104}\left((-1)^5 \begin{vmatrix} 4 & 7 \\ 2 & 2 \end{vmatrix} \right) = \frac{1}{104}\left(-(8-14) \right)$$

$$= \frac{6}{104} = \boxed{\frac{3}{52}}$$

Cramer's Rule

The adjugate formula has an application to linear equations. Specifically, recall that if A is an invertible matrix, then a system of equations

$$A\vec{x} = \vec{b}$$

has unique solution $\vec{x} = A^{-1}\vec{b}$. Applying the adjugate formula, this becomes

$$\vec{x} = \frac{1}{\det A}(\operatorname{adj} A)\,\vec{b}.$$

Multiplying this out, we obtain Cramer's Rule (see book for details of how to derive this).

Cramer's Rule: If A is an $n \times n$ invertible matrix, the solution to the system $A\vec{x} = \vec{b}$ is

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \ldots, \quad x_n = \frac{\det A_n}{\det A},$$

where for each k, A_k is the matrix obtained from A by replacing column k with $\vec{b}$.

Ex: $$\underbrace{\begin{bmatrix} 2 & 1 & 7 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{\vec{b}}$$

$\det A = 4 \begin{vmatrix} 1 & 7 \\ 0 & 1 \end{vmatrix} = 4$, so A is invertible.

$$\det A_1 = \begin{vmatrix} 0 & 1 & 7 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 7 \\ 0 & 1 \end{vmatrix} = 3$$

$\vec{b}$

expansion along row 3

$$\det A_2 = \begin{vmatrix} 2 & 0 & 7 \\ 3 & 0 & 1 \\ 4 & 3 & 0 \end{vmatrix} = -3 \begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix} = -3(2-21) = 57$$

expansion along col. 2

$$\det A_3 = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 0 \\ 4 & 0 & 3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = -3(3) = -9$$

expansion along row 2

so $x_1 = \frac{3}{4}$, $x_2 = \frac{57}{4}$, $x_3 = \frac{-9}{4}$

Note: Cramer's rule is <u>not</u> a good way to solve systems of equations Just like with the adjugate formula, it is only useful in theoretical applications.

Application: Polynomial interpolation

If we have several data points, how do we "fit" a polynomial to the data?

Ex: We are given the following stock price data:

Date	Stock price
Feb 1	\$1
Apr 1	\$4
July 1	\$6

We want to estimate the price of the stock on August 1 by finding a best fit quadratic polynomial:

$$p(x) = r_0 + r_1 x + r_2 x^2$$

where x = month of the year, $p(x)$ = price on the first of that month.

Our 3 data points are $p(2) = 1$, $p(4) = 4$, $p(7) = 6$.

$$\begin{aligned} r_0 + 2r_1 + 4r_2 &= 1 \\ r_0 + 4r_1 + 16r_2 &= 4 \\ r_0 + 7r_1 + 49r_2 &= 6 \end{aligned}$$

This is three linear equations in 3 variables, so we can solve:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 1 & 4 & 16 & 4 \\ 1 & 7 & 49 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 2 & 12 & 3 \\ 0 & 5 & 45 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & 6 & 3/2 \\ 0 & 5 & 45 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & 6 & 3/2 \\ 0 & 0 & 15 & -5/2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & 6 & 3/2 \\ 0 & 0 & 1 & -1/6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -8 & -2 \\ 0 & 1 & 6 & 3/2 \\ 0 & 0 & 1 & -1/6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10/3 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1/6 \end{array}\right]$$

So $p(x) = -\frac{10}{3} + \frac{5}{2}x - \frac{1}{6}x^2$.

So an estimate for August 1 is $p(8) = \frac{-10}{3} + \frac{5}{2}\cdot 8 + \frac{1}{6}\cdot 8^2 = 6$

Note: An n degree polynomial has $n+1$ coefficients, so if we have $n+1$ distinct data points, we can use this same method to try to find a degree n polynomial fit to the data. It turns out there is always a unique polynomial that works, called the interpolating polynomial. That is:

Theorem: If $a_1, \dots, a_n$ are distinct numbers, then

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} \neq 0.$$

This is called a Vandermonde determinant.

It has a formula given by:

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1 \le j < i \le n} (a_i - a_j).$$

Practice problems: 3.2 : 1bc, 2ade, 3, 6b, 8c, 9b, 22b